LAP and virtual levels of operators in Banach spaces
Application to Schrödinger operators in 2D

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Let $(-\Delta + W)\psi = z\psi, \quad x \in \mathbb{R}^2; \quad W$ is Δ-compact.

1. When is $z_0 = 0$ a "virtual level" of $-\Delta + W$?

2. If it is not, $(-\Delta + W - zI)^{-1}$: ¿¿¿ $\longrightarrow$???, uniformly bounded near z_0?

3. Is $z_0 = 0$ a "virtual level" **if and only if** there is $\psi \in L^\infty(\mathbb{R}^2) \setminus 0$?

4. What if W is nonselfadjoint and/or nonlocal?

1 Radiation principle and LAP

Helmholtz equation: $(-\Delta - z)u(x) = f(x) \in L^2(\mathbb{R}^3), \quad x \in \mathbb{R}^3, \quad z \in \mathbb{C}$

If $z \notin [0, +\infty)$: unique L^2-solution $u(x) = (-\Delta - zI)^{-1} f = \frac{e^{-|x|\sqrt{-z}}}{4\pi|x|} * f$.

If $z = k^2$, $k \geq 0$: could be no L^2-solution; a solution is not unique.

A way to specify a unique solution: "**radiation principle**"

- **Limiting absorption principle (LAP)** [Ignatowsky[05], Sveshnikov[50]]:

$$u(x) = \lim_{\varepsilon\to 0+} (-\Delta - (k + \mathrm{i}\boldsymbol{\varepsilon})^2 I)^{-1} f(x), \quad \text{so} \quad u(x) \sim \lim_{\varepsilon\to 0+} e^{+\mathrm{i}(k+\mathrm{i}\varepsilon)r}/r \sim e^{+\mathrm{i}kr}/r$$

- **Sommerfeld radiation condition** [Sommerfeld[12]]:

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0, \quad \text{so} \quad u(x) \sim e^{+\mathrm{i}kr}/r$$

Further development of LAP

- Limit of the resolvent at the essential spectrum in certain spaces, [Eidus[62], Vainberg[66], Rejto[69], Agmon[70]]:

$$\exists \lim_{\boldsymbol{z\to z_0>0},\ \operatorname{Im} z>0} (-\Delta - zI)^{-1}\colon\ L^2_s(\mathbb{R}^d) \to L^2_{-s}(\mathbb{R}^d), \quad \forall s>1/2, \quad \forall d\ge 1$$

$$\|u\|_{L^2_s(\mathbb{R}^d)} := \|(1+|x|)^s u\|_{L^2(\mathbb{R}^d)}$$

What if $z \to z_0 = 0$?

$d=3$, LAP [Nirenberg & Walker[73], Jensen & Kato[79]]:

$$\exists \lim_{z\to 0,\ z\notin[0,+\infty)} (-\Delta - zI)^{-1}\colon\ L^2_s(\mathbb{R}^3) \to L^2_{-s'}(\mathbb{R}^3), \quad \forall s,\ s'>1/2, \quad s+s'\ge 2$$

$d=1$, no LAP: $\quad (-\partial_x^2 - zI)^{-1} \sim \dfrac{e^{-|x-y|\sqrt{-z}}}{2\sqrt{-z}}, \quad \nexists$ limit as $z\to 0$

$d=2$, no LAP: $\quad (-\Delta - zI)^{-1} \sim -\ln|x-y| - \ln\sqrt{-z}, \quad \nexists$ limit as $z\to 0$

2 Virtual levels and virtual states

Singularity of the resolvent at threshold (usually for $-\Delta + V$ near $z_0 = 0$):

[Birman[61], Faddeev[63], Vainberg[68], Yafaev[74], Vainberg[75], Simon[76], Rauch[78]]...

$\boldsymbol{d = 3}$: [Yafaev[75], Jensen & Kato[79]] at most one virtual state "s"

$\boldsymbol{d = 2}$: [Bollé, Gesztesy, & Danneels[88]], [Jensen & Nenciu[01]]

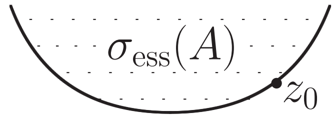
up to three virtual states: "s^1", "p^2"

boundedness of $|V|^{\frac{1}{2}}(-\Delta + V - zI)^{-1}|V|^{\frac{1}{2}}$

(weights are not optimal!)

Definition 1 (Virtual levels, virtual states).

[Boussaïd & Comech[21], Boussaïd & Comech[22]]



$z \in \Omega$

$$A \in \mathscr{C}(\mathbf{X}),\ \ \mathbf{E} \hookrightarrow \mathbf{X} \overset{\varphi}{\hookrightarrow} \mathbf{F},\ \ \Omega \subset \rho(A); \qquad \exists \hat{A} \in \mathscr{C}(\mathbf{F}), \quad \varphi \circ A = \hat{A} \circ \varphi$$

▶ $z_0 \in \sigma_{\mathrm{ess}}(A) \cap \partial\Omega$ **satisfies LAP relative to** $\mathbf{E}, \mathbf{F}, \Omega$ if

$$\exists\ (A - z_0 I)^{-1}_{\mathbf{E},\mathbf{F},\Omega} := \operatorname*{w-lim}_{z \to z_0,\, z \in \Omega} (A - zI)^{-1} : \mathbf{E} \to \mathbf{F}$$

▶ z_0 **virtual level relative to** $\mathbf{E}, \mathbf{F}, \Omega$ **if** $\exists\, B : \mathbf{F} \to \mathbf{E}$, A-compact, s.t.

$$\exists\ (A + B - z_0 I)^{-1}_{\mathbf{E},\mathbf{F},\Omega} := \operatorname*{w-lim}_{z \to z_0,\, z \in \Omega} (A + B - zI)^{-1} : \mathbf{E} \to \mathbf{F}$$

- $\exists\, B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$; smallest rank of such B: **rank of virtual level** z_0
- **virtual state:** $\Psi \in \operatorname{Range}\big((A + B - z_0 I)^{-1}_{\mathbf{E},\mathbf{F},\Omega}\big)$, $\quad \Psi \neq 0, \quad (\hat{A} - z_0)\Psi = 0$

Example 1. $N : \mathbf{X} \to \mathbf{X}, \ \ x \mapsto 0$. If $\dim \mathbf{X} = \infty$, then $(N + B - zI)^{-1}$ does not satisfy LAP relative to $\mathbf{E}, \mathbf{F}, \mathbb{C} \setminus \{0\}$ for any $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}, \ \ B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$

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Example 2. L, the left shift in $\ell^2(\mathbb{N})$, $\ \sigma(L)=\{|z|\le 1\}$,

$$L-zI=\begin{bmatrix}-z & 1 & 0 & \dots\\ 0 & -z & 1 & \dots\\ 0 & 0 & -z & \dots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix};\qquad (L-zI)^{-1}=-\begin{bmatrix}z^{-1} & z^{-2} & z^{-3} & \dots\\ 0 & z^{-1} & z^{-2} & \dots\\ 0 & 0 & z^{-1} & \dots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix},\quad |z|>1$$

satisfies LAP at any $|z_0|=1$ relative to $\ell^1(\mathbb{N}),\ell^\infty(\mathbb{N}),\{|z|>1\}$

Example 1. $N : \mathbf{X} \to \mathbf{X}, \ \ x \mapsto 0.$ If $\dim \mathbf{X} = \infty$, then $(N + B - zI)^{-1}$ does not satisfy LAP relative to $\mathbf{E}, \mathbf{F}, \mathbb{C} \setminus \{0\}$ for any $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}, \ B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$

Example 2. L, the left shift in $\ell^2(\mathbb{N}), \ \ \sigma(L) = \{|z| \le 1\}$,

$$L - zI = \begin{bmatrix} -z & 1 & 0 & \dots \\ 0 & -z & 1 & \dots \\ 0 & 0 & -z & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \qquad (L - zI)^{-1} = -\begin{bmatrix} z^{-1} & z^{-2} & z^{-3} & \dots \\ 0 & z^{-1} & z^{-2} & \dots \\ 0 & 0 & z^{-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \ \ |z| > 1$$

satisfies LAP at any $|z_0| = 1$ relative to $\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}), \{|z| > 1\}$

Example 3. $A = \frac{d}{dx}$ in $L^2(\mathbb{R}), \ \ \sigma(A) = \mathrm{i}\mathbb{R}$ $\qquad (\frac{d}{dx} - z)u = f(x)$

If $\operatorname{Re} z < 0, \ \left(\frac{d}{dx} - zI\right)^{-1} : f \mapsto u(x) = \int_{-\infty}^{x} e^{z(x-y)} f(y)\, dy, \ \ L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$

satisfies LAP at any $z_0 \in \mathrm{i}\mathbb{R}$ relative to $\boldsymbol{L^2_s}(\mathbb{R}), \boldsymbol{L^2_{-s}}(\mathbb{R}), \{\operatorname{Re} z < 0\}, \ s > 1/2$

Remarks:

- A virtual level may be an eigenvalue (if the corresponding Ψ is in $\mathbf{X}$)
- Virtual levels at endpoints are known as "threshold eigenvalues/resonances"
- "Critical" Schrödinger operators have virtual levels at thresholds

  [Simon[81], Murata[86], Gesztesy & Zhao[91], Pinchover & Tintarev[06]]
- Spectral singularities [Naimark[54], Schwartz[60], Pavlov[66], Ljance[67]]:

selfadjoint operators have no spectral singularities but could have virtual levels at threshold points.

- Not related to "ε-pseudospectrum" [Landau[75]]

Theorem 1 (LAP vs. virtual levels).

Let A satisfy LAP at $z_0 \in \sigma_{\mathrm{ess}}(A) \cap \partial\Omega$ relative to $\mathbf{E}, \mathbf{F}, \Omega$

If $B : \mathbf{F} \to \mathbf{E}$ is A-compact, then **either** $A + B$ satisfies LAP at z_0,
or z_0 is a virtual level (of rank $r \geq 1$) relative to $\mathbf{E}, \mathbf{F}, \Omega$

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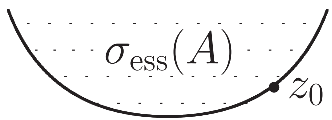
Theorem 2 (LAP vs. bifurcations from $\sigma_{\mathrm{ess}}(A)$).

Let $z_0 \in \sigma_{\mathrm{ess}}(A) \cap \partial\Omega$ be of finite rank $r \geq 0$ relative to $\mathbf{E}, \mathbf{F}, \Omega$

$\exists$ a (desired) bifurcation of a family of eigenvalues from z_0 into Ω under an arbitrarily small perturbation from $\mathscr{B}_0(\mathbf{F}, \mathbf{E})$ **iff** $\boldsymbol{r \geq 1}$



$z \in \Omega \subset \rho(A)$

$\bullet\ z_j \in \sigma_{\mathrm{p}}(A + B_j)$

$$z_j \to z_0, \quad \|B_j\|_{\mathbf{F}\to\mathbf{E}} \to 0 \quad \text{as } j \to \infty$$

Theorem 3 (Resolvent from codimension-1 resolvent).

Let $A \in \mathscr{C}(\mathbf{X})$, $z_0 \in \sigma(A) \cap \partial\Omega$.

Assume $\exists\, P = P^2 \in \mathscr{B}(\mathbf{E})$, $\exists\, Q = Q^2 \in \mathscr{B}(\mathbf{F})$, $\operatorname{rank} P = \operatorname{rank} Q = 1$,

$(A - zI)^{-1}(I - P)$, $(I - Q)(A - zI)^{-1}$ satisfy LAP at z_0 relative to $\mathbf{E}, \mathbf{F}, \Omega$

Then z_0 is a virtual level of A of rank 1 relative to $\mathbf{E}, \mathbf{F}, \Omega$

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Then z_0 is a virtual level of A of rank 1 relative to $\mathbf{E}, \mathbf{F}, \Omega$

Proof: Let $P = \phi\langle\xi, \cdot\rangle, \quad Q = \Psi_0\langle\eta, \cdot\rangle;$ denote $\boldsymbol{B} = \boldsymbol{\phi}\langle\boldsymbol{\eta}, \cdot\rangle : \mathbf{F} \to \mathbf{E}$

$\exists\, \Psi_z \in \mathbf{F}$: $\Psi_z \xrightarrow{\mathbf{F}} \Psi_0, \; r_z := (A - z)\Psi_z \xrightarrow{\mathbf{E}} 0; \quad (A + B - z)\Psi_z = r_z + \phi\langle\eta, \Psi_z\rangle$

- To solve $(A + B - z)u = g \in \mathbf{E}$:

$v_z := (A - zI)^{-1}(I - P)g \quad \Rightarrow \quad (A + B - z)v_z = (I - P)g + \phi\langle\eta, v_z\rangle$

$w_z := (A - zI)^{-1}(I - P)r_z \quad \Rightarrow \quad (A + B - z)w_z = (I - P)r_z + \phi\langle\eta, w_z\rangle$

$\Rightarrow \quad u = c_1 v_z + c_2 w_z + c_3 \Psi_z \in \mathbf{F} \qquad c_1,\, c_2,\, c_3 \in \mathbb{C}$ uniformly bounded in z

3 Application to Schrödinger operators in 2D

Theorem 4 (LAP vs. virtual levels in 2D).

Let $W : L^2_{-1,-\nu}(\mathbb{R}^2) \to L^2_{1,\mu}(\mathbb{R}^2)$ be Δ-compact, $\mu,\ \nu > \frac{1}{2},\ \ \mu + \nu > 2,\ \boldsymbol{\nu} \le \mathbf{\frac{3}{2}}$

Either there is LAP for $R_W(z) = (-\Delta + W - zI)^{-1}$,

$$(-\Delta + W - zI)^{-1} : L^2_{1,\mu}(\mathbb{R}^2) \to L^2_{-1,-\nu}(\mathbb{R}^2), \quad z \to 0, \quad z \in \rho(-\Delta + W)$$

or $\exists \Psi \in L^2_{-1,-\nu}(\mathbb{R}^2), \qquad \Psi \ne 0, \qquad (-\Delta + W)\Psi = 0;$

moreover, if $W : f(x) \mapsto V(x) f(x),\ |V(x)| \le C\langle x\rangle^{-2-0}$, then $\Psi \in L^\infty(\mathbb{R}^2)$

$$\|\psi\|_{L^2_{s,\mu}} = \|\langle x\rangle^s \big(\ln(1 + \langle x\rangle)\big)^\mu \psi\|_{L^2}$$

Example 4. $\Psi = \big(\ln(1 + \langle x\rangle)\big)^{1-0}$ can be a virtual state: let $\rho \in C_0^\infty,\ \ \langle \rho, \Psi\rangle = 1$

$$W := \Delta\Psi\langle \rho,\ \cdot\ \rangle \ \ \Rightarrow \ \ (-\Delta + W)\Psi = 0, \ \ \Psi \in \textbf{Range}\,\big((-\Delta + B - z_0 I)^{-1}\big)$$

Resolvent from codimension-1 resolvent in 2D

Consider $R_0(z) = (-\Delta - zI)^{-1}$ in $L^2(\mathbb{R}^2)$, $\quad z \in \mathbb{C} \setminus [0, +\infty)$;

$$R_0(x, y; z) = \frac{\mathrm{i}}{4} H_0^{(1)}(|x-y|\sqrt{z}) = -\frac{1}{2\pi} \ln \sqrt{-z} - \frac{1}{2\pi} \ln |x-y| + O(1)$$

$$\Rightarrow \qquad (-\Delta - zI)^{-1} \overbrace{(1 - \Phi\langle 1, \cdot \rangle)}^{I-P} : L^2_{1,\mu}(\mathbb{R}^2) \to L^2_{-1,-\nu}(\mathbb{R}^2), \qquad \int_{\mathbb{R}^2} \Phi \, dx = 1$$

satisfies LAP on **zero mean functions** at $z_0 = 0$ $\qquad (\mu,\ \nu > \frac{1}{2},\ \mu + \nu > 2)$

$\Rightarrow$ If $B : L^2_{-1,-\nu}(\mathbb{R}^2) \to L^2_{1,\mu}(\mathbb{R}^2)$ is Δ-compact and $-\Delta + B$ has no virtual levels, then

$$(-\Delta + B - zI)^{-1} : L^2_{1,\mu}(\mathbb{R}^2) \to L^2_{-1,-\nu}(\mathbb{R}^2) \qquad \text{satisfies LAP at } z_0 = 0$$

Example 5.

Recall: left shift L in $\ell^2(\mathbb{N})$ satisfies LAP relative to $\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}), \{|z| > 1\}$

Let $A = L + \boldsymbol{e}_1\langle \boldsymbol{e}_1, \cdot\rangle$; then $(A - I)\boldsymbol{e}_1 = 0$

Clearly, $z_0 = 1$ is a virtual level [eigenvalue] of rank 1.

There is no projection $P \in \mathscr{B}_{00}(\ell^1(\mathbb{N}))$ such that

$$(A - zI)^{-1}(I - P) : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$$

would be uniformly bounded near $z_0 = 1$ (so no limit as $z \to z_0$)

Similarly, **there is no projection** $Q \in \mathscr{B}_{00}(\ell^\infty(\mathbb{N}))$ such that

$$(I - Q)(A - zI)^{-1} : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$$

would be uniformly bounded near $z_0 = 1$ (so no limit as $z \to z_0$)

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