

Functional model for generalised resolvents and its application to time-dispersive media

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Joint work with Yulia Ershova, Sergey Naboko

Mathematical aspects of the physics with non-self-adjoint operators

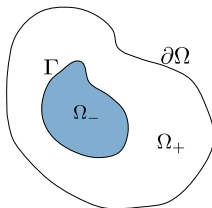
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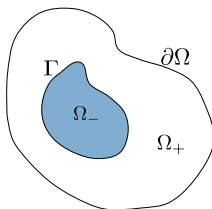


Basic operator setup

Smooth bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, a simply connected $C^{1,1}$ inclusion $\Omega_- \subset \Omega$ at a positive distance from $\partial\Omega$:



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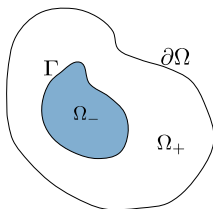
Consider $\tilde{H} = L^2(\Omega_+) \oplus \mathbb{C}$ and

$$\text{dom}(A) = \left\{ \begin{pmatrix} u_+ \\ \beta \end{pmatrix} \in \tilde{H} : u_+ \in H^2(\Omega_+), u_+|_{\Gamma} = \frac{\beta}{\sqrt{|\Omega_-|}} \mathbb{1}_{\Gamma}, \left. \frac{\partial u_+}{\partial n_+} \right|_{\partial\Omega} = 0 \right\}$$

($u|_{\Gamma}$ is the trace, $\mathbb{1}_{\Gamma}$ is the unity function on Γ , and n_+ is the exterior normal to $\partial\Omega$).

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($u|_{\Gamma}$ is the trace, $\mathbb{1}_{\Gamma}$ is the unity function on Γ , and n_+ is the exterior normal to $\partial\Omega$).

On $\text{dom}(A)$, set

$$A \begin{pmatrix} u_+ \\ \eta \end{pmatrix} = \begin{pmatrix} -\Delta u_+ \\ \frac{1}{\sqrt{|\Omega_-|}} \int_{\Gamma} \frac{\partial u_+}{\partial n_+} \end{pmatrix}.$$

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where $\gamma > 0$, $\eta \in \mathbb{R}$, $\omega \in \mathbb{C}$, $|\omega| = 1$, and

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The pair $(u, \beta)^\top$ describes the approximation of the solution to a second-order DE with contrasting parameters in a "resonant" asymptotic regime: u and β correspond to the leading-order behaviour on the "soft" (resonant) and "stiff" parts of the medium.

Reformulation as generalised resolvent

A is the **Neumark-Štraus dilation** ("out-of-space extension") for the solution operator $\mathcal{R}(z)$ of

$$\begin{aligned} -\partial_\tau^* \partial_\tau u - zu &= f, \\ u(0) = \omega u(l), \quad Du &= (\gamma - \eta^2 z)u(0). \end{aligned} \tag{*}$$

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$$A \begin{pmatrix} u \\ \beta \end{pmatrix} - z \begin{pmatrix} u \\ \beta \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

and the projection $P_{\tilde{H}}$ onto $\tilde{H} := L^2(0, l) \oplus \{0\}$.

On the abstract level,
$$\mathcal{R}(z) = P_{\tilde{H}}(A - zI)^{-1}|_{\tilde{H}},$$

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$\mathcal{R}(z)$ is a **generalised resolvent** – Neumark (1940), Štraus (1954).

In (*) the spectral parameter z is in equation and the boundary conditions:

$$\tilde{A}_{\max} u - zu = f, \quad \tilde{\Gamma}_1 u = B(z)\tilde{\Gamma}_0 u, \quad u \in W^{2,2}(0, l),$$

- \tilde{A}_{\max} is generated by $\partial_\tau^* \partial_\tau$ on $W^{2,2}(0, l)$
- $\tilde{\Gamma}_0, \tilde{\Gamma}_1 : W^{2,2}(0, l) \rightarrow \mathbb{C}^2$ satisfy

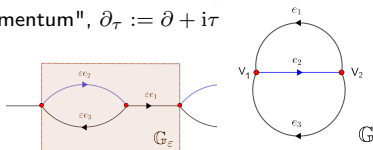
$$\langle \tilde{A}_{\max} u, v \rangle_{L^2(0, l)} - \langle u, \tilde{A}_{\max} v \rangle_{L^2(0, l)} = \langle \tilde{\Gamma}_1 u, \tilde{\Gamma}_0 v \rangle_{\mathbb{C}^2} - \langle \tilde{\Gamma}_0 u, \tilde{\Gamma}_1 v \rangle_{\mathbb{C}^2},$$
- $-B(z)$ is an operator-valued R -function.

Example: Homogenisation of graphs with contrast

Operator family $\{A^\varepsilon\}_{\varepsilon>0}$ in $\bigoplus_{\mathbb{Z}} \bigoplus_{j=1}^3 L_2(0, \varepsilon l_j)$ generated by $-a^\varepsilon \partial^2$, with ε -periodic a^ε . Suppose $a^\varepsilon = a_j > 0$ on edges e_j , $j = 1, 2$ and $a_2 = \varepsilon^2$.

"Fibre operator" A_τ^ε is generated by $\varepsilon^{-2} a^\varepsilon \partial_\tau^* \partial_\tau$

$\tau \in [-\pi, \pi)$ "quasimomentum", $\partial_\tau := \partial + i\tau$

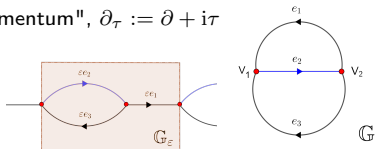


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$$\eta = \sqrt{l_1 + l_3}, \quad \gamma = (l_1 a_1^{-1} + l_3 a_3^{-1})^{-1} (\tau/\varepsilon)^2,$$

$$\omega = -\bar{\xi}_\tau / |\xi_\tau|, \quad \xi_\tau := -l_1^{-1} a_1 e^{i\tau(l_1+l_3)} - l_3^{-1} a_3 e^{-i\tau l_2}$$

$$\text{dom}(\mathcal{A}_{\text{hom}}^\tau) = \{(u, \beta)^\top \in W^{2,2}(0, l_2) \oplus \mathbb{C} : u(0) = \omega u(l_2) = \eta^{-1} \beta\}$$

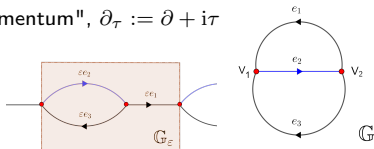
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Denote $H := \bigoplus_{j=1}^3 L^2(0, l_j)$. SIAP (2019):

$$\| (A_\tau^\varepsilon - z)^{-1} - \Psi^* (\mathcal{A}_{\text{hom}}^\tau - z)^{-1} \Psi \|_{H \rightarrow H} = C\varepsilon^2,$$

$\Psi : H \rightarrow L^2(0, l_2) \oplus \mathbb{C}$ is a partial isometry.

Boundary triple framework (finite deficiency indices)

Suppose that:

- A_{\max} is the adjoint to a densely defined symmetric operator A_{\min} on a separable Hilbert space H ("physical region space")
- Γ_0, Γ_1 are linear mappings of $\text{dom}(A_{\max}) \subset H$ to a separable Hilbert space \mathcal{H} ("boundary space")

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A. $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is called a *boundary triple* for A_{\max} (Kořubei, 1975, based on ideas of M. Krein) if:

- $\forall u, v \in \text{dom}(A_{\max})$ one has

$$\langle A_{\max}u, v \rangle_H - \langle u, A_{\max}v \rangle_H = \langle \Gamma_1 u, \Gamma_0 v \rangle_{\mathcal{H}} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{\mathcal{H}}$$

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B. The operator-valued Herglotz function $m = m(z)$, defined by

$$m(z)\Gamma_0u_z = \Gamma_1u_z, \quad u_z \in \ker(A_{\max} - z), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-,$$

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C. An extension A_B (such that $A_{\min} \subset A_B \subset A_{\max}$) is called *almost solvable* if there is a boundary triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for A_{\max} and a bounded operator B on \mathcal{H} such that

$$(u \in \text{dom}(A_B) \iff \Gamma_1 u = B\Gamma_0 u) \quad \forall u \in \text{dom}(A_{\max})$$

$$\text{dom}(A_{\min}) = \left\{ \begin{pmatrix} u \\ \beta \end{pmatrix} \in W_2^2(0, l) \oplus \mathbb{C} : u(0) = \omega u(l) = \eta^{-1} \beta, Du = 0 \right\},$$

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Theorem

A. The triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where

$$\mathcal{H} := \mathbb{C}, \quad \Gamma_0 \begin{pmatrix} u \\ \beta \end{pmatrix} := Du, \quad \Gamma_1 \begin{pmatrix} u \\ \beta \end{pmatrix} := \frac{\beta}{\eta} - u(0), \quad \begin{pmatrix} u \\ \beta \end{pmatrix} \in W_2^2(0, l) \oplus \mathbb{C},$$

is a boundary triple for A_{\max} .

B. The M -function of A_{\max} with respect to $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is given by

$$m(z) = -\frac{\sin \sqrt{z}l}{2k(\Re(e^{i\tau l} \bar{\omega}) - \cos \sqrt{z}l)} - \frac{1}{\eta^2 z - \gamma}.$$

2-component ("spectral") form of Pavlov's dilation

Consider $A_{\min} \subset A_i \subset A_{\max}$: $\text{dom}(A_i) := \{f \in \text{dom}(A_{\max}) : \Gamma_1 f = i\Gamma_0 f\}$.

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"Characteristic function" of A_i : $s(z) = (m(z) - i)(m(z) + i)^{-1}$, $z \in \mathbb{C}_+$.

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"Characteristic function" of A_i : $\mathfrak{s}(z) = (m(z) - i)(m(z) + i)^{-1}$, $z \in \mathbb{C}_+$.

- The space $\mathfrak{H} := L^2\left(\mathbb{C}^2; \begin{pmatrix} 1 & \bar{\mathfrak{s}} \\ \mathfrak{s} & 1 \end{pmatrix}\right)$

is the completion of the linear set of functions $(\tilde{g}, g)^\top : \mathbb{R} \rightarrow \mathbb{C}^2$ in the norm

$$\int_{\mathbb{R}} \left\langle \begin{pmatrix} 1 & \overline{\mathfrak{s}(s)} \\ \mathfrak{s}(s) & 1 \end{pmatrix} \begin{pmatrix} \tilde{g}(s) \\ g(s) \end{pmatrix}, \begin{pmatrix} \tilde{g}(s) \\ g(s) \end{pmatrix} \right\rangle_{\mathbb{C}^2} ds$$

- "Incoming" and "outgoing" subspaces of \mathfrak{H} :

$$D_- := \begin{pmatrix} 0 \\ H_-^2 \end{pmatrix}, \quad D_+ := \begin{pmatrix} H_+^2 \\ 0 \end{pmatrix}$$

- The orthogonal projection P_K onto

$$K := \mathfrak{H} \ominus (D_- \oplus D_+) = \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathfrak{H} : g_- \in H_-^2, g_+ \in H_+^2 \right\}$$

is given by

$$P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+ g_- \\ g - P_- g_+ \end{pmatrix},$$

where P_{\pm} are the orthogonal Riesz projections in $L^2(\mathbb{R})$ onto H_{\pm}^2 .

Naboko-type functional model for Neumark-Štraus dilation

There is a self-adjoint operator \mathcal{A} ("dilation" – Pavlov, 1975) in $L^2(\mathbb{R}_-) \oplus H \oplus L^2(\mathbb{R}_+)$ and an isometry $\Phi : L^2(\mathbb{R}_-) \oplus H \oplus L^2(\mathbb{R}_+) \rightarrow \mathfrak{H}$ such that

$$P_H(\mathcal{A} - zI)^{-1}|_H = (A_i - zI)^{-1}, \quad z \in \mathbb{C}_-,$$

$$\Phi(\mathcal{A} - zI)^{-1} = (\cdot - z)^{-1}\Phi,$$

$$\Phi|_H H = K,$$

and the following statement holds (Ryzhov, 2008 – inspired by Naboko, 1980):

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❶ If $z \in \mathbb{C}_- \cap \rho(A_0)$ and $(\tilde{g}, g)^\top \in K$, then

$$\Phi(A_0 - zI)^{-1}\Phi^* \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \frac{1}{\cdot - z} \begin{pmatrix} \tilde{g} \\ g - (\overline{\mathfrak{s}(z)} + 1)^{-1}g_-(z) \end{pmatrix}.$$

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Here, $g_\mp(z)$ are the values at z of the analytic continuations of $g_- := \tilde{g} + \overline{\mathfrak{s}}g \in H_-^2$ and $g_+ := \mathfrak{s}\tilde{g} + g \in H_+^2$ into the corresponding half-plane.

Clarke measure associated with m and spectral model on \mathbb{R}

Define μ to be the Nevanlinna representation of the Herglotz function $-m^{-1}$:

$$-m(\lambda) = C_0 + C_1\lambda - \int_{\mathbb{R}} \left(\frac{1}{\lambda - \sigma} + \frac{\sigma}{1 + \sigma^2} \right) d\mu(\sigma)$$

where $C_0, C_1 \in \mathbb{C}$.

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Theorem

A. H is isometric to $L_2(\mathbb{R}, \pi d\mu)$. This isometry is the composition of (Kiselev, Naboko, 2006)

$$H \ni v \mapsto -\frac{1}{\sqrt{\pi}} \Gamma_0(A_i - \cdot)^{-1} v = g_+ \in K_s := H_+^2 \ominus sH_+^2$$

and the embedding of K_s into $L_2(\mathbb{R}, \pi d\mu)$. The latter is realised by taking the boundary values on the real line of functions in K_s , which exist μ -a.e.

B. Under the above isometry, $(A_0 - z)^{-1}$ is unitarily equivalent to the multiplication by $(\cdot - z)^{-1}$ in $L_2(\mathbb{R}, \pi d\mu)$.

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μ is a "Clark measure" (Clark, 1972)

Clark measures were introduced in the context of the shift operator, with further function-theoretic insight by Alexandrov (1993)

Application to high-contrast homogenisation

μ is a counting measure with masses at $\lambda = \lambda_j$ solving

$$\cos \sqrt{\lambda_j} l - (\eta^2 \lambda_j - \gamma) \frac{\sin \sqrt{\lambda_j} l}{2\sqrt{\lambda_j}} = \Re(e^{i\tau l} \bar{\omega}), \quad j = 1, 2, \dots \quad (**)$$

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$$\mu(\{\lambda_j\}) = \frac{2(\eta^2 \lambda_j - \gamma)^2}{\eta^2 + \frac{\gamma}{\lambda_j} + 2l + \frac{l}{\sqrt{\lambda_j}} (\eta^2 \lambda_j - \gamma) \cot \sqrt{\lambda_j} l}$$

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When $a_1 = a_3 = a$, $l_1 = l_3 = (1 - l_2)/2$, the equation (**) takes a more compact form:

$$\cos \sqrt{\lambda_j} l_2 - \left\{ (1 - l_2) \lambda_j - \frac{a}{1 - l_2} \left(\frac{\tau}{\varepsilon} \right)^2 \right\} \frac{\sin \sqrt{\lambda_j} l_2}{2\sqrt{\lambda_j}} = |\cos \tau|, \quad j = 1, 2, \dots$$

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A corollary is a "spectral" representation for the generalised resolvent (in the form of an explicit pseudodifferential operator)

$$\mathcal{R}_{\text{hom}}^\tau(z) = \mathcal{P}(\mathcal{A}_{\text{hom}}^\tau - z)^{-1} \Big|_{L^2(0, l_2)}$$

for which $\mathcal{A}_{\text{hom}}^\tau$ is the Neumark-Štraus dilation.

Here \mathcal{P} is the natural orthogonal projection of $L^2(0, l_2) \oplus \mathbb{C}$ onto $L^2(0, l_2)$.