The generalized Birman-Schwinger principle

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Mathematical aspects of the physics with non-self-adjoint operators

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PART I

The classical Birman-Schwinger principle

Laplacians and Schrödinger operators

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- More assumptions on $V \Rightarrow$ more/better properties of $\sigma(H)$

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Theorem [Cuenin, Ibrogimov 2021]

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$$\lambda_0 \in \mathbb{C} \setminus \mathbb{R} \text{ eigenvalue of } H \;\Rightarrow\; \sqrt{2}|\lambda_0| \leq \sqrt{|\lambda_0| + |\mathrm{Re}\,\lambda_0|}\,\|V\|_{L^1}$$

A few references

Classical works:

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Some non-selfadjoint contributions:

- Gestzesy, Latushkin, Mitrea, and Zinchenko '05
- Frank, Laptev, Lieb, and Seiringer '06
- Demuth, Hansmann, and Katriel '09
- Cuenin, Laptev, and Tretter '14
- Enblom '16
- Frank, Laptev, and Safronov '16
- Cuenin '17
- Frank and B. Simon '17
- Cuenin and Siegl '18
- Fanelli, Krejčiřík and Vega '19
- Cassano, Ibrogimov, Krejčiřík and Štampach '20
- Hansmann and Krejčiřík '22

PART II

An abstract Birman-Schwinger principle

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Assumptions

H_0 closed operator in $\mathcal{H}$ with $\rho(H_0) \neq \emptyset$

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PROBLEM: BS-principle for generalized ev's in $\ker(H - \lambda_0)^k$

PART III

The Birman-Schwinger principle for generalized eigenvectors (Jordan chains)

Jordan chains of functions and operators

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Definition [Keldysh '51, Markus'88]

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Jordan chains of functions and operators

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M operator function defined on $\Omega \subset \mathbb{C}$ and dom $M(\lambda) = \mathcal{D}$. If

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PART IV

Robin realisations of elliptic PDO's

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More precisely, A_B is the m-sectorial operator associated to

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which is closed and sectorial with $\operatorname{dom} \mathfrak{a} = H^1(\Omega)$.

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Recall $\lambda \mapsto D(\lambda)$ holomorphic operator function on $\rho(A_D)$.

Jordan chains and the Birman-Schwinger principle

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and $f_{-1} = 0$.

The special case $k = 0$

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Corollary

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- If $D(\lambda_0)\varphi_0 = B\varphi_0$ and $\varphi_0 \neq 0$ then unique solution $f_0 \in H^1(\Omega)$ of
$$(\mathcal{A} - \lambda_0) f_0 = 0, \qquad f_0|_{\mathcal{C}} = \varphi_0,$$
is an eigenvector of A_B at λ_0.

Thank you for your attention