Quantum Control by Moving Walls

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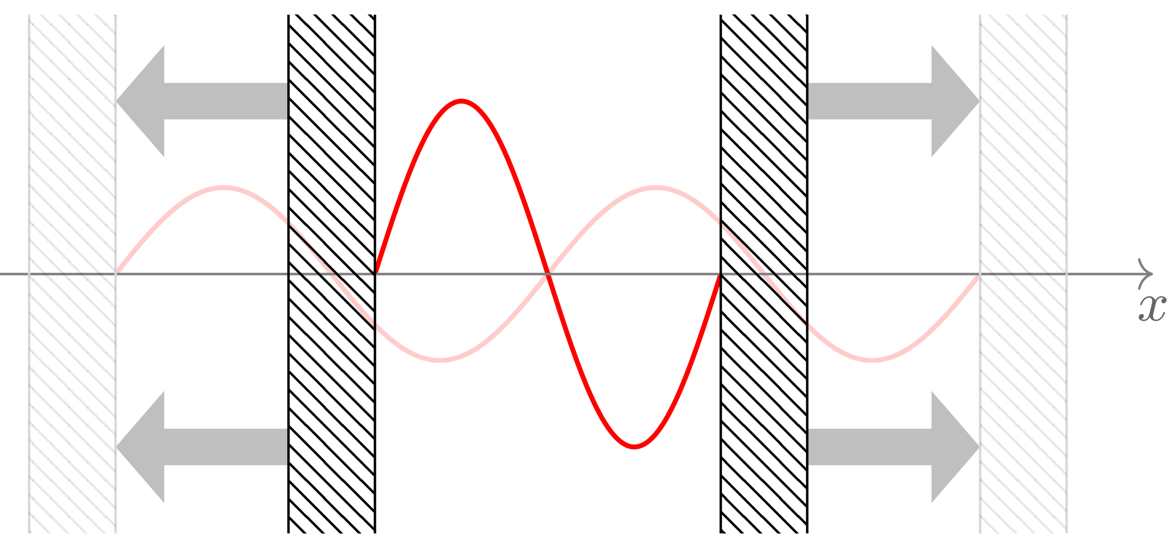
Can we control a particle in a box only by moving its walls?

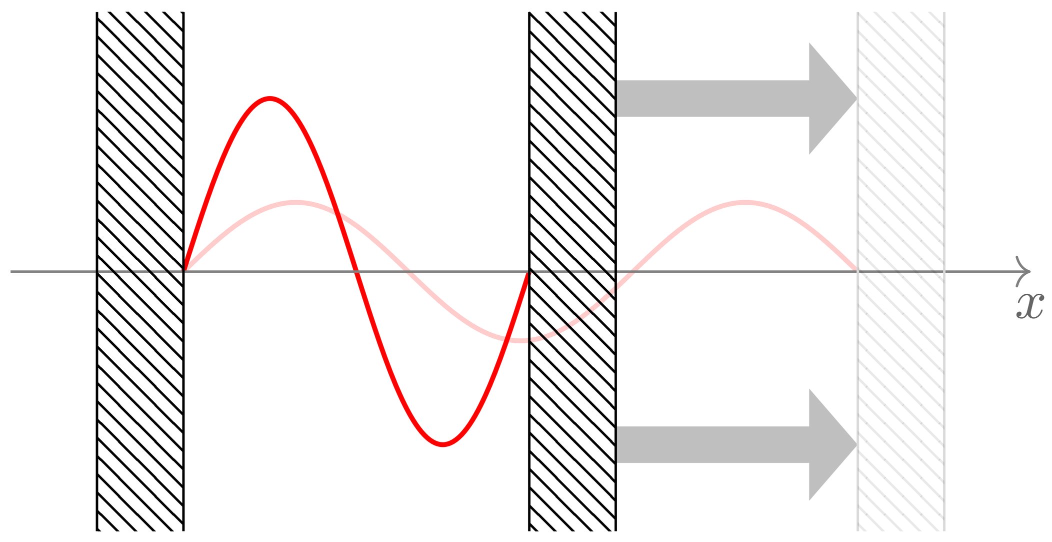
- Consider a quantum particle inside a 1D box:

$$\Omega_{\ell,d} = \left[d - \frac{\ell}{2}, d + \frac{\ell}{2}\right]$$

- Can we transform any state into another by
  - moving only the right wall of the box?
  - moving both walls symmetrically?



Case 1: Moving both walls symmetrically.



Case 2: Moving only the right wall.

Outline

- The problem has t-dependent domain!
  - ⇝ Does it even have well-defined dynamics?
- Transform into an equivalent problem with constant (form) domain, (Σ).
  - ⇝ Well-defined dynamics!
- Introduce an auxiliar system with known controllability properties.
  - ⇝ Bilinear quantum control system.
- Use stability (Thm. 2) to transfer controllability properties to the original system.

Form-linear Hamiltonians [1, 2]

- Given a self-adjoint t-dependent Hamiltonian $H(t)$ such that $\langle \Phi, H(t)\Psi\rangle > m$ for all t and $\Phi, \Psi \in \operatorname{dom} H(t)$, there is a unique sesquilinear form h_t satisfying

$$h_t(\Psi, \Phi) = \langle \Psi, H(t)\Phi\rangle \qquad \forall \Psi \in \operatorname{dom} h_t, \forall \Phi \in \operatorname{dom} H(t).$$

- When $\operatorname{dom} h_t = \mathcal{H}^+$ for all t, we say that $H(t)$ is a form-linear Hamiltonian if

$$h_t(\Psi, \Phi) = \sum_{i=0}^{N} f_i(t) h^{(i)}(\Psi, \Phi), \quad \text{with} \quad |h^{(i)}(\Phi, \Phi)| < K\big(h_0(\Phi, \Phi) + (m+1)\|\Phi\|\big).$$

- **Theorem 1 (Well-defined dynamics)** *If f_i are piecewise C^1, for every $\Phi \in \mathcal{H}^+$ there is a (weak) solution $\Phi(t)$ of the Schrödinger equation for $H(t)$ with $\Phi(0) = \Phi$.*
- **Theorem 2 (Stability)** *For a family $h_{n,t}(\Psi, \Phi) = \sum_{i=0}^{N} f_{n,i}(t) h^{(i)}(\Psi, \Phi)$ of form-linear Hamiltonians with piecewise C^2 functions $f_{n,i}$ uniformly bounded, if $\sup_n \|f'_{n,i}\|_{L^1} < \infty$ and $\{f_{i,n}\}_n$ converges to $f_{i,0}$ in L^1, then the solutions $\Phi_n(t)$ converge to $\Phi_0(t)$ in $\mathcal{H}$.*

Bilinear quantum control

- Space of states: Hilbert space $\mathcal{H}$.
- Control $u(t) \in (0, r)$.
- Controlled evolution: $\Phi(t) \in \mathcal{H}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -i\left(H_0 + u(t)H_1\right)\Phi(t).$$

- **Approximately controllable** if for any tolerance $\varepsilon > 0$ and initial and target states, $\Phi_0, \Phi_1 \in \mathcal{H}$, there is T and $u(t)$ such that the evolution $\Phi(t)$ satisfies

$$\Phi(0) = \Phi_0 \quad \text{and} \quad \|\Phi(T) - \Phi_1\| < \varepsilon.$$

Moving walls control setting

- We consider as control the real function $u(t)$ and set $d(t) = d_0 + \delta u(t)$, $\ell(t) = \ell_0 + u(t)$.
- As controlled Hamiltonian, we take $H(t) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L^2(\Omega_{\ell(t),d(t)})$ with Dirichlet B.C.
- The isometries $(T(t)\Psi)(x) = \sqrt{\ell(t)}\Psi(\ell(t)x + d(t))$ transform the moving walls problem into another with fixed domain, since:
  - For any t, $T(t)L^2(\Omega_{\ell(t),d(t)}) = L^2(\Omega_{1,0})$.
  - If $\Psi \in \operatorname{dom} H(t)$, then $\Phi(t) = T(t)\Psi(t)$ satisfies Dirichlet B.C. in $\Omega_{1,0}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -i\left[\frac{1}{\ell(t)^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\dot\ell(t)}{2\ell(t)}(xp + px) - \frac{\dot d(t)}{\ell(t)}p\right]\Phi. \qquad (\Sigma)$$

- This system (Σ) has a form-linear Hamiltonian.

Auxiliar system in $L^2(\Omega_{1,0})$

- $$\tilde H(t) = \frac{1}{\ell_0}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{v(t)}{\ell_0}\left(\frac{1}{2}(xp + px) - \delta p\right).$$
- Bilinear quantum control system.
  - ⇝ Approximately controllable with piecewise constant controls by [4], see details in [3, Thm. 4.2].

Approximate controllability by moving walls

Theorem 3 ([3, Theorem 2.9]) *For any $r > 0$, it holds:*

- *If $\Delta\ell\Delta d \neq 0$ or $\Delta\ell = \Delta d = 0$, the moving walls control system is approximately controllable with piecewise afine control $u(t)$ satisfying $u'(t) < r$.*
- *If $\Delta d = 0$, the moving walls control system is approximately controllable with piecewise afine control $u(t)$ satisfying $u'(t) < r$ between spaces with defined parity.*

Idea of the proof: Let $v : [0, T] \to (0, r)$ be the control for the auxiliary system, divide $[0, T]$ in n pieces and build controls $u_n(t)$ for (Σ) as depicted in Fig. 1. Applying Thm. 2 yields the result.

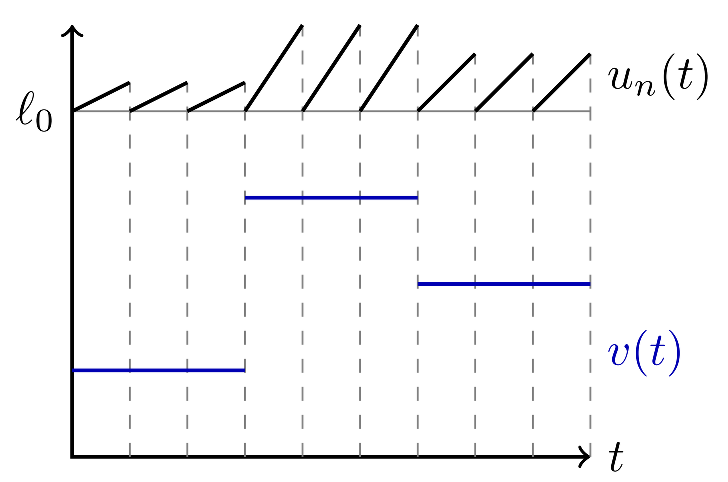


Figure 1: Construction of $u_n(t)$, $n = 9$.

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[3] A. Balmaseda, D. Lonigro, and J.M. Pérez-Pardo. On Global Approximate Controllability of a Quantum Particle in a Box by Moving Walls. *SIAM Journal on Control and Optimization*, 62(2):826–852, 2024.

[4] U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti. A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule. *Communications in Mathematical Physics*, 311(2):423–455, 2012.

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