Curved quantum nonlinear waveguides

Laura Baldelli
IMAG, University of Granada
labaldelli@ugr.es

joint work with David Krejčiřík
(Czech Technical University in Prague)

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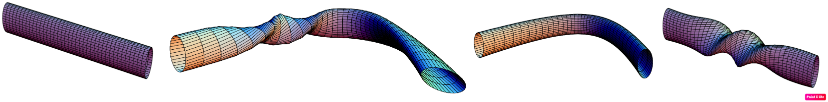
L.B., Krejčiřík *arXiv:2312.10357*

We investigate the influence of the geometric deformations of tubes Ω in $\mathbb{R}^d$ $d \geq 1$, non-compact and non-complete manifold, on the **nonlinear Dirichlet p-Laplacian** $-\Delta_p$ with $p \in (1, \infty)$

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

In particular, we proved that:

- bent and asymptotically straight tubes leads to discrete eigenvalues below the essential spectrum,
  Chenaud, Duclos, Freitas, Krejčiřík *Diff. Geom. Appl.* (2005)
- twisted tubes generates Hardy-type inequalities.
  Krejčiřík *Amer. Math. Soc., Providence, RI* (2008)



Results on the linear Dirichlet Laplacian $-\Delta$

- Bending is attractive, i.e. the spectrum is pushed down leading existence of discrete eigenvalues below the essential spectrum

Exner, Šeba *Phys. Rev. B* (1989) $2D$

Goldstone, Jaffe *J. Math. Phys.* (1992) $3D$

Duclos, Exner *Rev. Math. Phys* (1995) $3D$

Chenaud, Duclos, Freitas, Krejčiřík *Diff. Geom. Appl.* (2005) arbitrary dimensions adapting Frenet frame

Krejčiřík, Šediváková *Rev. Math. Phys.* (2012) arbitrary dimensions with optimal regularity hypotheses (adapted frame)

Relying on the quantum-mechanical motivation, the spectral result can be illustratively interpreted as that the particle in a curved quantum waveguide gets trapped.

- Twisting is repulsive, i.e. it raises the spectral threshold, geometrically eliminating the discrete eigenvalues, by establishing a Hardy inequality for the Dirichlet Laplacian in twisted three-dimensional tubes

  Ekholm, Kovařík, Krejčiřík *Arch. Ration. Mech. Anal.* (2008)

  A more robust technique to derive the geometrically induced Hardy inequalities was later developed in

  Krejčiřík *Amer. Math. Soc., Providence, RI* (2008)

  Krejčiřík, Zuazua *J. Math. Pures Appl.* (2010)

  Roughly, twisting prevents the particle from being trapped in the waveguide.

Krejčiřík *lecture notes (Mini-courses in Mathematical Analysis, Padova, Italy, June 2023)* `https://hal.science/hal-04159525`

Let $\Omega \subset \mathbb{R}^d$ open, $d \geq 1$, we introduce the *spectral threshold* and the *essential spectral threshold* by the variational formulae

$$\lambda_1(\Omega) := \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p \, dx}{\displaystyle\int_\Omega |u(x)|^p \, dx}, \qquad \lambda_\infty(\Omega) := \sup_{K \Subset \Omega} \lambda_1(\Omega \setminus K).$$

In general, $\lambda_1(\Omega)$ and $\lambda_\infty(\Omega)$ extend the variational characterisations of the lowest point in the spectrum (Rayleigh–Ritz) and the essential spectrum (Persson) of the self-adjoint Dirichlet Laplacian $-\Delta$ in $L^2(\Omega)$.

- If Ω is bounded, then the infimum is achieved by a function $u \in W_0^{1,p}(\Omega)$ called the *first eigenfunction* (or *ground state*) of Ω and the spectral threshold $\lambda_1(\Omega)$ is also called the *first* (or *principal*) *eigenvalue* of Ω. Moreover, $\lambda_\infty(\Omega) = \infty$.

  Lindqvist *Proc. Amer. Math. Soc.* (1990)

- If Ω is unbounded, then the existence of the ground state is a highly non-trivial property.

Das, Pinchover, Devyver *arXiv:2303.03527*

The relatively parallel adapted frame

Let $\Gamma : \mathbb{R} \to \mathbb{R}^d$ be a $C^{1,1}$-smooth unit-speed curve. Then $T := \Gamma'$ is a unit tangent vector field along Γ and $\kappa := |\Gamma''|$ is the (locally bounded) curvature function of Γ. There exist (a.e. differentiable) unit normal vector fields $N_1, \dots, N_{d-1} : \mathbb{R} \to \mathbb{R}^d$ such that

$$\begin{pmatrix} T \\ N_1 \\ \vdots \\ N_{d-1} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \dots & \kappa_{d-1} \\ -\kappa_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -\kappa_{d-1} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ \vdots \\ N_{d-1} \end{pmatrix},$$

where $\kappa_i : \mathbb{R} \to \mathbb{R}$ are (locally bounded) s.t. $\kappa_1^2 + \cdots + \kappa_{d-1}^2 = \kappa^2$.

Then $(T, N_1, \dots, N_{d-1})$ is the *relatively parallel adapted frame*.

The relatively parallel adapted frame VS Frenet frame

The relatively parallel adapted frame differs from the *Frenet frame* which requires an extra (classically C^d-smoothness) regularity of Γ, and it excluded curves with vanishing curvature somewhere.

Krejčiřík, Zahradová *Oper. Matrices* (2020)

Curvilinear coordinates

Consider a one-parametric family of differentiable rotation matrices

$$R : \mathbb{R} \to \mathsf{SO}(d-1), \qquad R' \in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{(d-1)\times(d-1)}).$$

Let us introduce the mapping $\mathscr{L} : \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^d$ defined by

$$\mathscr{L}(s,t) := \Gamma(s) + t_\mu\, R_{\mu\nu}(s)\, N_\nu, \quad (s,t) \in \Omega_0 = \mathbb{R}\times\omega, \quad \Omega_{\kappa,R} = \mathscr{L}(\Omega_0),$$

where $\omega \subset \mathbb{R}^{d-1}$ is the cross-section of the tube.
Assume $\Omega_{\kappa,R}$ does not overlap itself, which means $\kappa \in L^\infty(\mathbb{R})$ and

$$a\,\|\kappa\|_{L^\infty(\mathbb{R})} < 1 \quad \text{with} \quad a := \sup_{t\in\omega} |t|.$$

Let us think of $\Omega_{\kappa,R}$ as the Riemannian manifold (Ω_0, g) where the induced metric $g := (\nabla\mathscr{L}) \cdot (\nabla\mathscr{L})^T$ has $\det(g) := f^2$ with

$$f(s,t) := \ 1 - t_\alpha R_{\alpha\beta}(s)\kappa_\beta(s).$$

$\mathscr{L}$ is a local $C^{0,1}$-diffeomorphism and its Jacobian satisfies

$$0 < 1 - a\,\|\kappa\|_{L^\infty(\mathbb{R})} \le f(s,t) \le 1 + a\,\|\kappa\|_{L^\infty(\mathbb{R})} < \infty, \quad (s,t) \in \Omega_0.$$

Passing to the curvilinear coordinates, $\psi := u \circ \mathscr{L}$, we have

$$\lambda_1(\Omega_{\kappa,R}) = \inf_{\substack{u \in W_0^{1,p}(\Omega_{\kappa,R}) \\ u \neq 0}} \frac{\|\nabla u(x)\|_p^p}{\|u(x)\|_p^p} \rightsquigarrow \lambda_1(\Omega_{\kappa,R}) = \inf_{\substack{\psi \in W_0^{1,p}(\Omega_0,g) \\ \psi \neq 0}} \frac{Q[\psi]}{\|\psi\|^p} =: \lambda_1(\Omega_0, g).$$

where $W_0^{1,p}(\Omega_0, g)$ is the closure of $C_0^\infty(\Omega_0)$ with respect to the norm $(Q[\psi] + \|\psi\|^p)^{1/p}$, with $\|\psi\| := \left(\int_{\Omega_0} |\psi(s,t)|^p f(s,t)\, ds\, dt\right)^{1/p}$ and

$$Q[\psi] := \int_{\Omega_0} \left(\left| \frac{(\partial_s - f_\mu(s,t)\partial_{t_\mu})\psi(s,t)}{f(s,t)} \right|^2 + |\nabla_t \psi(s,t)|^2 \right)^{p/2} f(s,t)\, ds\, dt.$$

Assuming $a\,\|\kappa\|_{L^\infty(\mathbb{R})} < 1$, we have two possibilities

- $\mathscr{L}$ injective, thus $\mathscr{L} : \Omega_0 \to \Omega_{\kappa,R}$ is a global diffeomorphism and (Ω_0, g) an embedded submanifold.
- $\mathscr{L}$ a local $C^{0,1}$-diffeomorphism and interpret (Ω_0, g) as an immersed submanifold.

The essential spectral threshold could have two interpretations

$$\lambda_\infty(\Omega_{\kappa,R}) = \sup_{K \Subset \Omega_{\kappa,R}} \lambda_1(\Omega_{\kappa,R} \setminus K) \qquad \lambda_\infty(\Omega_{\kappa,R}) = \sup_{K \Subset \Omega_0} \lambda_1(\Omega_0 \setminus K, g).$$

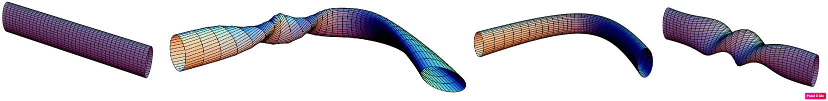
Tubes

Recalling $\Omega_0 = \mathbb{R} \times \omega$ with cross-section $\omega \subset \mathbb{R}^{d-1}$ and a general *bent twisted* tube about Γ is defined as

$$\Omega_{\kappa,R} = \mathscr{L}(\Omega_0) = \left\{\Gamma(s) + t_\mu \, R_{\mu\nu}(s)\, N_\nu : \ (s,t) \in \Omega_0\right\},$$

then,

- if $\kappa = 0 = R'$, then $\Omega_{\kappa,R} (= \Omega_{0,I} = \Omega_0)$ is straight.
- if $\kappa \neq 0 = R'$, then $\Omega_{\kappa,R}$ is nontrivially bent and untwisted.
- if $\kappa = 0 \neq f_\mu \partial_{t_\mu} \phi_1 (\Rightarrow R' \neq 0)$, then $\Omega_{\kappa,R}$ is unbent and nontrivially twisted, where $f_\mu(s,t) := t_\alpha R'_{\alpha\beta}(s) R_{\mu\beta}(s)$ and ϕ_1 is the first eigenfunction of ω.
- if $\lim_{|s|\to\infty} \kappa(s) = 0$ and $\lim_{|s|\to\infty} \int_\omega |f_\mu(s,t)\partial_{t_\mu}\phi_1(t)|^p \, dt = 0$ then $\Omega_{\kappa,R}$ is asymptotically straight.



The hypothesis $f_\mu \partial_{t_\mu} \phi_1 \neq 0$

- This assumption on the asymmetry of ω w.r.t. R, is considered to avoid that $\Omega_{0,R}$ is congruent to Ω_0, even if $R' \neq 0$.
- If $d = 3$, the condition is equivalent to a simultaneous validity of the following two requirements:

$$\theta' \neq 0 \qquad \text{and} \qquad \omega \text{ is not circular},$$

where

$$R(s) = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix},$$

is a a convenient parameterization and $\theta : \mathbb{R} \to \mathbb{R}$ is a differentiable function with locally bounded derivative.
- If $d \geq 4$, the situation is more complicated because we cannot separate the "longitudinal" and "transverse" variables from the condition. Anyway, a sufficient condition is:

$$R' \neq 0 \qquad \text{and} \qquad 0 \notin \omega$$

Theorem 0: Stability for straight tubes

$\lambda_1(\Omega_0) = \lambda_1(\omega)$, where $\Omega_{\kappa,R} = \Omega_0 = \mathbb{R} \times \omega$.

Brasco *Bruno Pini Math. Anal. Semin.* (2018) General product sets $\Omega = \mathbb{R}^{d-k} \times \omega$, with $\omega \subset \mathbb{R}^k$ open bounded, $k \in \{1, \dots, d-1\}$

Since ω is assumed to be bounded and connected, $\lambda_1(\omega) > 0$ is a simple eigenvalue of the Dirichlet p-Laplacian in ω.

More specifically, there exists a unique (up to a constant multiple) positive minimiser $\phi_1 \in W_0^{1,p}(\omega)$ of

$$\lambda_1(\Omega_{\kappa,R}) = \inf_{\substack{u \in W_0^{1,p}(\Omega_{\kappa,R}) \\ u \neq 0}} \frac{\|\nabla u(x)\|^p_{L^p(\Omega_{\kappa,R})}}{\|u(x)\|^p_{L^p(\Omega_{\kappa,R})}}$$

We choose it normalised to 1 in $L^p(\omega)$, i.e., $\int_\omega |\phi_1(t)|^p \, dt = 1$.

Lindqvist *Proc. Amer. Math. Soc.* (1990)

Sketch of the proof of Theorem 0

Recalling that

$$\lambda_1(\Omega_0) = \inf_{\substack{\psi \in W_0^{1,p}(\Omega_0, g) \\ \psi \neq 0}} \frac{Q[\psi]}{\|\psi\|^p},$$

- First, by the Poincaré inequality and Fubini theorem, we proved that

$$Q[\psi] = \int_{\Omega_0} \left(|\partial_s \psi|^2 + |\nabla_t \psi|^2\right)^{p/2} ds\, dt \geq \lambda_1(\omega) \|\psi\|^p,$$

- To prove the opposite inequality, define

$$\varphi_n(s) = 1 \text{ if } |s| \leq n, \quad \varphi_n(s) = 2 - \frac{|s|}{n} \text{ if } |s| \in (n, 2n), \quad \varphi_n(s) = 0 \text{ if } |s| \geq 2n$$

and $\psi_n(s,t) := \varphi_n(s)\phi_1(t) \in W_0^{1,p}(\Omega_0)$, so that

$$R[\psi_n] := \frac{Q[\psi_n] - \lambda_1(\omega)\|\psi_n\|^p}{\|\psi_n\|^p} \xrightarrow[n \to \infty]{} 0.$$

For any $a, b, q \geq 0$ and $\alpha, \beta > 0$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, one has

$$(a+b)^q \leq \alpha^q a^q + \beta^q b^q.$$

$-\Delta_p - \lambda_1(\omega)$ in Ω_0 is critical

Instability of $-\Delta_p$ with respect to small perturbations

Let $V \in C_0^\infty(\Omega_0)$ be non-positive and non-trivial. Then

$$\lambda_1^V(\Omega_0) := \inf_{\substack{\psi \in W_0^{1,p}(\Omega_0) \\ \psi \neq 0}} \frac{\displaystyle\int_{\Omega_0} |\nabla \psi|^p \, ds \, dt + \int_{\Omega_0} V |\psi|^p \, ds \, dt}{\|\psi\|^p}$$

$$< \inf_{\substack{\psi \in W_0^{1,p}(\Omega_0) \\ \psi \neq 0}} \frac{\displaystyle\int_{\Omega_0} |\nabla \psi|^p \, ds \, dt}{\|\psi\|^p} = \lambda_1(\Omega_0) = \lambda_1(\omega)$$

The last but one equality is the definition of $\lambda_1(\Omega_0)$, the last one is given by Theorem 0. The main claim is the strict inequality.

We proved that, for n sufficiently large, then

$$Q_1^V[\psi_n] := \int_{\Omega_0} |\nabla\psi_n|^p \, ds\, dt + \int_{\Omega_0} V\, |\psi_n|^p \, ds\, dt - \lambda_1(\omega) \int_{\Omega_0} |\psi_n|^p \, ds\, dt < 0,$$

where $\psi_n(s,t) := \varphi_n(s)\phi_1(t) \in W_0^{1,p}(\Omega_0)$, with φ_n defined as before and ϕ_1, as above, is the eigenfunction associated to $\lambda_1(\omega)$.

Remark

We have to split the proof into two cases because of the nature of the nonlinearity: $p \leq 2$ and $p > 2$,

For any $a, b, q \geq 0$ and $\alpha, \beta > 0$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, one has

$$(a+b)^q \leq \alpha^q a^q + \beta^q b^q.$$

In particular, if $0 \leq q \leq 1$ then $(a+b)^q \leq a^q + b^q$.

Subcriticality/Supercriticality/Criticality of $-\Delta_p - \lambda_1(\omega)$ in $\Omega_{\kappa,R}$

If $\Omega_{0,R}$ is unbent non-trivially twisted, then $-\Delta_p - \lambda_1(\omega)$ in $\Omega_{0,R}$ is ***subcritical*** in the sense of existence of a Hardy inequality $\underline{-\Delta_p - \lambda_1(\omega) \geq \rho}$ with a positive function ρ.

The shifted operator $-\Delta_p - \lambda_1(\omega)$ in straight tubes Ω_0 is ***critical*** in the sense that no such positive function ρ exists, namely the spectral threshold of $-\Delta_p - \lambda_1(\omega) + V$ in Ω_0 is negative whenever the perturbation $V \in C_0^\infty(\Omega_0)$ is non-positive and non-trivial.

In non-trivially bent untwisted tubes, the operator $-\Delta_p - \lambda_1(\omega)$ in $\Omega_{\kappa,I}$ may be understood as ***supercritical***, because $\lambda_1(\Omega_{\kappa,I}) - \lambda_1(\omega)$ is negative even if $V = 0$.

Cazacu, Krejčiřík, Lam, Laptev *Nonlinearity* (2024)

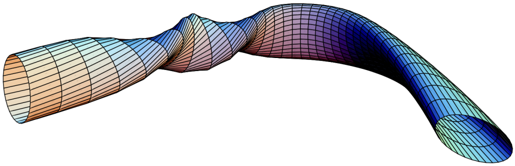
Theorem 1: Stability for asymptotically straight tubes

If $\Omega_{\kappa,R}$ is asymptotically straight, namely

$$\lim_{|s|\to\infty} \kappa(s) = 0 \qquad \text{and} \qquad \lim_{|s|\to\infty} \int_\omega |f_\mu(s,t)\partial_{t_\mu}\phi_1(t)|^p \, dt = 0$$

holds, then $\lambda_\infty(\Omega_{\kappa,R}) = \lambda_1(\omega)$. In particular, for straight tubes, then $\lambda_\infty(\Omega_0) = (\lambda_1(\omega) =)\lambda_1(\Omega_0)$: essential spectral gap is zero.

A sufficient condition to ensure the validity of the second limit is that $R'(s) \to 0$ as $|s| \to \infty$.



Krejčiřík, Kříž *Publ. RIMS, Kyoto University* (2005)

Krejčiřík, Lu *J. Math. Phys.* (2014)

The lower bound holds for tubes merely asymptotically unbent.

Lower bound

If $\lim_{|s|\to\infty} \kappa(s) = 0$, then $\lambda_\infty(\Omega_{\kappa,R}) \geq \lambda_1(\omega)$.

While, the upper bound holds for asymptotically untwisted tubes.

Upper bound

If $\lim_{|s|\to\infty} \kappa(s) = 0$ and $\lim_{|s|\to\infty} r(s) := \lim_{|s|\to\infty} \int_\omega |f_\mu(s,t)\partial_{t_\mu}\phi_1(t)|^p \, dt = 0$

holds, then $\lambda_\infty(\Omega_{\kappa,R}) \leq \lambda_1(\omega)$.

Sketch of the proof of the lower bound

Note that

$$\lambda_\infty(\Omega) := \sup_{K \Subset \Omega} \lambda_1(\Omega \setminus K) \geq \lambda_1(\Omega \setminus K)$$

for any "trial" compact subset K of Ω_0. For any positive numbers ε (small) and l (large), we set

$$K := [-l, l] \times \overline{\omega_\varepsilon} \qquad \text{with} \qquad \omega_\varepsilon := \{t \in \omega : \operatorname{dist}(t, \partial\omega) > \varepsilon\}.$$

Let $\psi \in W_0^{1,p}(\Omega_0 \setminus K, g)$ be arbitrary and consider

$$Q[\psi] \geq \int_{\Omega_0} |\nabla_t \psi|^p \, f \, ds \, dt = \int_{[-l,l]} \int_{\omega \setminus \overline{\omega_\varepsilon}} |\nabla_t \psi|^p \, f \, dt \, ds + \int_{\mathbb{R} \setminus [-l,l]} \int_{\omega} |\nabla_t \psi|^p \, f \, dt \, ds$$

$$\geq \cdots \geq \min \left\{ \frac{1 - a \, \|\kappa\|_{L^\infty(\mathbb{R})}}{1 + a \, \|\kappa\|_{L^\infty(\mathbb{R})}} \, \lambda_1(\omega \setminus \overline{\omega_\varepsilon}), \frac{1 - a \, \|\kappa\|_{L^\infty(\mathbb{R} \setminus [-l,l])}}{1 + a \, \|\kappa\|_{L^\infty(\mathbb{R} \setminus [-l,l])}} \, \lambda_1(\omega) \right\} \|\psi\|^p.$$

From $\lambda_1(\omega \setminus \overline{\omega_\varepsilon}) \to \infty$ as $\varepsilon \to 0$ and $\lim_{|s| \to \infty} \kappa(s) = 0$, we conclude.

Chorwadwala, Mahadevan, Toledo *ESAIM: COCV* (2015)
Faber–Krahn inequality for the Dirichlet p-Laplacian

Sketch of the proof of the upper bound

Fix any $K \Subset \Omega_0$ and let us define $\psi_n(s,t) := \tilde{\varphi}_n(s)\phi_1(t)$ with $\tilde{\varphi}_n(s) := \varphi_n(s-n^2)$, where φ_n is the sequence already defined. Note that $\tilde{\varphi}_n$ is "localised at" ∞ meaning that $\inf \operatorname{supp} \tilde{\varphi}_n = n^2 - 2n \to \infty$ as $n \to \infty$. This ensures $\psi_n \in W_0^{1,p}(\Omega_0 \setminus K)$ for n sufficiently large, so

$$\frac{Q[\psi_n]}{\|\psi_n\|^p} \leq \frac{\alpha^{p/2}\,\tilde{\alpha}^p}{(1-a\,\|\kappa\|_\infty)^p}\,\frac{\int_{\mathbb{R}} |\varphi_n'|^p\,ds}{\int_{\mathbb{R}} |\varphi_n|^p\,ds} + \frac{\alpha^{p/2}\,\tilde{\beta}^p\,\|r\|_{n,\infty}}{(1-a\,\|\kappa\|_{n,\infty})^p} + \beta^{p/2}\,\frac{1+a\,\|\kappa\|_{n,\infty}}{1-a\,\|\kappa\|_{n,\infty}}\,\lambda_1(\omega).$$

By the properties of φ_n and $\kappa(s) \to 0$ and $r(s) \to 0$ as $s \to \infty$, then

$$\lim_{n\to\infty} \frac{Q[\psi_n]}{\|\psi_n\|^p} \leq \beta^{p/2}\,\lambda_1(\omega), \qquad \beta > 1$$

In summary, given any $K \Subset \Omega_0$ and any $\varepsilon > 0$, we have proved that

$$\lambda_1(\Omega_0 \setminus K, g) \leq \lambda_1(\omega) + \varepsilon \;\Rightarrow\; \lambda_\infty(\Omega_{\kappa,R}) \leq \lambda_1(\omega) + \varepsilon.$$

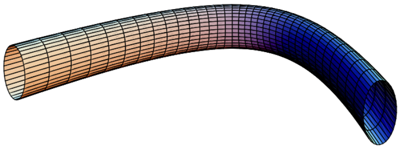
Since ε can be made arbitrarily small, the desired claim follows.

Theorem 2: Reduction of the spectral threshold

If $\Omega_{\kappa,I}$ is nontrivially bent, untwisted and ω is circular, i.e. $\omega = B_a(0)$ or $\omega = B_a(0) \setminus \overline{B_{a_0}(0)}$, $0 < a_0 < a$, then

$$\lambda_1(\Omega_{\kappa,I}) < \lambda_1(\omega) = \lambda_1(\Omega_0).$$

Chenaud, Duclos, Freitas, Krejčiřík *Diff. Geom. Appl.* (2005)



Corollary: Positivity of the essential spectral gap

If $\Omega_{\kappa,I}$ is nontrivially bent, asymptotically straight and ω is circular, then $\lambda_1(\Omega_{\kappa,I}) < \lambda_\infty(\Omega_{\kappa,I})$.

Without loss of generality, we may take $R = I$.

By the positivity and uniqueness of ϕ_1, it is possible to conclude that ϕ_1 is radially symmetric in balls and spherical shells.

- **Anoop, Bobkov, Sasi** *Trans. Amer. Math. Soc.* (2018)
- **Bhattacharya** *Proc. Amer. Math. Soc.* (1988)

Using this observation, one immediately has the following property

Lemma

If ω is circular, then

$$\int_\omega |\phi_1(t)|^p \, t \, dt = 0 = \int_\omega |\nabla \phi_1(t)|^p \, t \, dt.$$

Note that in the linear case $p = 2$, it follows by integrating by parts.

The symmetric assumption: ω is circular

This hypothesis on ω was employed in the use of the identity

$$\int_\omega |\nabla\phi_1(t)|^p \, t \, dt = \lambda_1(\omega) \int_\omega |\phi_1(t)|^p \, t \, dt. \tag{1}$$

In particular, (1) holds for circular domains due to the symmetry.

By using the test function $\varphi(t) := t\,\phi_1(t)$ in the weak formulation of the eigenvalue equation $-\Delta_p\phi_1 = \lambda_1(\omega)|\phi_1|^{p-2}\phi_1$ in ω, subject to Dirichlet boundary conditions $\phi_1 = 0$ on $\partial\omega$, the identity (1) is equivalent to

$$\int_\omega |\nabla\phi_1(t)|^{p-2} \, \nabla|\phi_1|^2 \, dt = 0. \tag{2}$$

Integrating by parts, (2) holds for any domain ω whenever $p = 2$.

For arbitrary $p \in (1, \infty)$, instead of assuming that ω is circular, Theorem 2 works for any domain ω satisfying (2).

Sketch of the proof of Theorem 2

Recalling that

$$\lambda_1(\Omega_{\kappa,I}) = \inf_{\substack{\psi \in W_0^{1,p}(\Omega_0,g) \\ \psi \neq 0}} \frac{Q[\psi]}{\|\psi\|^p},$$

the claim is equivalent to the existence of a (trial) function $\psi \in W_0^{1,p}(\Omega_0)$ for which

$$Q_1[\psi] := Q[\psi] - \lambda_1(\omega)\,\|\psi\|^p < 0.$$

- First step: take a regularisation of $\phi_1(t)$, again by using $\psi_n(s,t) := \varphi_n(s)\phi_1(t)$ and achieve an asymptotic equality instead of the strict inequality, after having splitting the proof into the two cases $p \leq 2$ and $p > 2$, namely

$$Q_1[\psi_n] \to 0 \text{ as } n \to \infty.$$

- Second step: perturb ψ_n by $\psi_{n,\varepsilon}$ in such a way that the strict inequality $Q_1[\psi_{n,\varepsilon}] < 0$ is achieved for n large and ε small.

Entering into details of the second step, we define

$$\psi_{n,\varepsilon} := \psi_n + \varepsilon\,\phi \qquad \text{with} \qquad \phi(s,t) := j(s)\,\xi(t)\,\phi_1(t), \qquad \text{where}$$

- → $\varepsilon \in \mathbb{R}$ and $j \in C_0^\infty(\mathbb{R})$ and $\xi \in L^\infty(\omega)$.
- → consider n large that $\varphi_n = 1$ on $\operatorname{supp} j$.
- → assume $|\varepsilon| \leq \varepsilon_0$, $\varepsilon_0 \in \mathbb{R}$ small s.t. $\varepsilon_0\,\|j\|_{L^\infty(\mathbb{R})}\,\|\xi\|_{L^\infty(\omega)} < 1$.
  Thus $\psi_{n,\varepsilon}(s,t) = \phi_1(t)(1+\varepsilon j(s)\xi(t)) > 0$ if $s \in \operatorname{supp} j$.

Let us write

$$Q_1[\psi_{n,\varepsilon}] = I_1(\varepsilon) - \lambda_1(\omega)\,I_2(\varepsilon) =: I(\varepsilon) \qquad \text{with} \qquad \begin{aligned} I_1(\varepsilon) &:= Q[\psi_{n,\varepsilon}], \\ I_2(\varepsilon) &:= \|\psi_{n,\varepsilon}\|^p. \end{aligned}$$

Our strategy is to employ the Taylor expansion

$$I(\varepsilon) = I(0) + I'(0)\,\varepsilon + o(\varepsilon) \qquad \text{as} \qquad \varepsilon \to 0,$$

so that $I(0) = Q_1[\psi_n] \to 0$ as $n \to \infty$, from the first step, while the remainder $o(\varepsilon)$ and $I'(0)$ is n-independent and, by choosing ε sufficiently small and of suitable sign, it is possible to ensure that $I'(0)\,\varepsilon + o(\varepsilon) < 0$. Then we choose n so large that also $I(\varepsilon) < 0$.

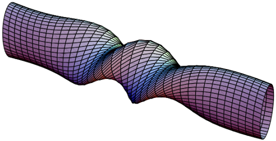
Theorem 3: Hardy inequality

If $\Omega_{0,R}$ is unbent, nontrivially twisted, then there exists a continuous function $\rho : \Omega_{0,R} \to \mathbb{R}^+$ s.t. for any $u \in W_0^{1,p}(\Omega_{0,R})$,

$$\int_{\Omega_{0,R}} |\nabla u|^p \, dx - \lambda_1(\omega) \int_{\Omega_{0,R}} |u|^p \, dx \geq \int_{\Omega_{0,R}} \rho \, |u|^p \, dx.$$

Equivalently, there exists a positive continuous function $\varrho : \Omega_0 \to \mathbb{R}$ such that the Hardy inequality holds for any $\psi \in W_0^{1,p}(\Omega_0, g)$

$$Q[\psi] - \lambda_1(\omega) \, \|\psi\|^p \geq \int_{\Omega_0} \varrho \, |\psi|^p \, ds \, dt.$$



Krejčiřík *Amer. Math. Soc., Providence, RI* (2008)

Krejčiřík, Zuazua *J. Math. Pures Appl.* (2010)

Define

$$\lambda_1^N(\Omega_0^l) := \inf_{\substack{\psi \in W_0^{1,p}(\Omega_0) \\ \psi \neq 0}} \frac{Q^l[\psi]}{\|\psi\|^{l,p}}$$

for any $\Omega_0^l := (-l, l) \times \omega$ with $l > 0$, where

$$Q^l[\psi] := \int_{\Omega_0^l} \left(\left| (\partial_s - f_\mu \partial_{t_\mu}) \psi \right|^2 + |\nabla_t \psi|^2 \right)^{p/2} ds\, dt, \quad \|\psi\|^{l,p} := \int_{\Omega_0^l} |\psi|^p \, ds\, dt.$$

Lemma

$$\lambda_1^N(\Omega_0^l) > \lambda_1(\omega) \qquad \Longleftrightarrow \qquad f_\mu \partial_{t_\mu} \phi_1 \neq 0 \quad \text{in} \quad \Omega_0^l.$$

By the Poincarè inequality $\lambda_1^N(\Omega_0^l) \geq \lambda_1(\omega)$. Moreover,

- $\Leftarrow$ By choosing the trial function $\psi(s,t) := \phi_1(t)$, it follows that $\lambda_1^N(\Omega_0^l) = \lambda_1(\omega)$ if $f_\mu \partial_{t_\mu} \phi_1 = 0$ in Ω_0^l.
- $\Rightarrow$ Assume that $f_\mu \partial_{t_\mu} \phi_1 \neq 0$ but $\lambda_1^N(\Omega_0^l) = \lambda_1(\omega)$. By compactness, the infimum is achieved by $\psi_1 \in W_0^{1,p}(\Omega_0) \restriction \Omega_0^l$. Since $\lambda_1^N(\Omega_0^l)$ is simple... we conclude that $f_\mu \partial_{t_\mu} \phi_1 = 0$ in Ω_0^l.

Kawohl, Lindqvist *Analysis* (2006)

Sketch of the proof of Theorem 3

1. "Local" Hardy inequality
   Given any $\psi \in W_0^{1,p}(\Omega_0, g)$, $l > 0$, by Poincarè inequality, one has

$$
\begin{aligned}
Q[\psi] - \lambda_1(\omega)\,\|\psi\|^p &= \int_{\Omega_0} \left[\left(\left|(\partial_s - f_\mu \partial_{t_\mu})\psi\right|^2 + |\nabla_t \psi|^2 \right)^{p/2} - \lambda_1(\omega)\,|\psi|^p \right] ds\,dt \\
&\geq \int_{\Omega_0^l} \left[\left(\left|(\partial_s - f_\mu \partial_{t_\mu})\psi\right|^2 + |\nabla_t \psi|^2 \right)^{p/2} - \lambda_1(\omega)\,|\psi|^p \right] ds\,dt \\
&= Q^l[\psi] - \lambda_1(\omega) \int_{\Omega_0^l} |\psi|^p \geq (\lambda_1^N(\Omega_0^l) - \lambda_1(\omega)) \int_{\Omega_0^l} |\psi|^p,
\end{aligned}
$$

   where $c_l := \lambda_1^N(\Omega_0^l) - \lambda_1(\omega) > 0$. This establishes a Hardy inequality with a non-negative and non-trivial function

$$
\varrho_l := c_l\, \chi_{\Omega_0^l}.
$$

2. "Global" Hardy inequality
   By a standard argument of partition of unity subordinated to a finitely local covering, we get a Hardy inequality with a positive ϱ.

   Pinchover, Tintarev *Calc. Var. Partial Differ. Equ.* (2007)

Future projects

- Replacing Dirichlet boundary conditions by *Neumann* and *Robin* boundary conditions
  - **Kovařík, Pankrashkin** *Calc. Var. Partial Differ. Equ* (2017)

  Indeed, apart from two-dimensional studies or thin-width asymptotics
  - **Krejčiřík** *ESAIM: Control, Optim. and Calc. of Var.* (2009)
  - **de Oliveira, Rossini** *Commun. Anal. Geom.* (2022)

  a detailed spectral-geometric analysis of higher-dimensional Robin waveguides remains open even in the linear case $p = 2$.

- Adding *magnetic* fields to the p-Laplacian
  - **Cazacu, Krejčiřík, Lam, Laptev** *Nonlinearity* (2024)
  - **Krejčiřík, Raymond** *Ann. H. Poincare* (2014)

Thank you!