Energy decay of solutions of the wave equation with unbounded damping at infinity

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Overview

- Introduction
- Motivation
- Main Result
- Remarks
- Methods

Introduction - The DWE with unbounded damping

The damped wave equation

$$\begin{cases} \partial_t^2 u(t,x) + 2a(x)\partial_t u(t,x) = \Delta u(t,x), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^d, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (f_1, f_2) \end{cases}$$

- t time variable, x space variable, Ω open set
- $0 \leq a \in L^1_{\mathrm{loc}}(\Omega)$ (a damping) - assumed to hold throughout
- 2nd order Cauchy problem → 1st order one via the matrix operator

$$G = \begin{pmatrix} 0 & I \\ \Delta & -2a \end{pmatrix}$$

- traditionally, focus on $a \in L^\infty(\Omega)$ (extensive literature)
- recent work on $a \notin L^\infty(\Omega)$ (Sobajima and Wakasugi; Ikehata and Takeda; Freitas, Siegl and Tretter; Arifoski and Siegl; Arnal; Freitas, Hefti and Siegl; Kleinhenz and Wang)
  - ◇ we focus on singularity of a at ∞, e.g. $\Omega = \mathbb{R},\ a(x) = x^2$
- Q: impact on spectral behaviour of G, long time decay of solutions $u(t,x)$?

Motivation - Polynomial decay of solutions

Theorem 1 ([Ikehata and Takeda, 2020])

Let $d \geq 3$, let $a \in C(\mathbb{R}^d)$ s.t. $a(x) \geq a_0 > 0$, $x \in \mathbb{R}^d$. Assume that the initial data $f = (f_1, f_2)^t$ satisfy

- $f_1 \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $f_2 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, and
- $af_1 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Then there exists a unique weak solution u satisfying

$$\|u(t,\cdot)\| \leq CI_{00}(f)(1+t)^{-1/2}, \qquad \|\partial_t u(t,\cdot)\| + \|\nabla u(t,\cdot)\| \leq CI_{00}(f)(1+t)^{-1}$$

where $C > 0$ and

$$I_{00}(f)^2 = \|f_1\|^2 + \|\nabla f_1\|^2 + \|af_1\|_{L^1}^2 + \|af_1\|^2 + \|f_2\|^2 + \|f_2\|_{L^1}^2$$

Observations

- *e.g.* $a(x) = (1+|x|^2)^{\frac{\alpha}{2}}$, $\alpha \geq 0$, or $a(x) = e^{|x|}$
  - ◇ $\sigma_{\text{ess}}(G) = (-\infty, 0] \rightsquigarrow$ polynomial decay of solutions
- the proof relies on constructing approximate weak solutions of the Cauchy problem using a multiplier method (modified Morawetz)
- low dimensions $d = 1, 2$ not covered
- Q: can we extend this result?

Main Result - Energy and L^2-norm decay of solutions

Theorem 2 (A., Gerhat, Royer, Siegl)
Assume that $a(x) \geq a_0 > 0$ a.e. in Ω and let the initial data $f = (f_1, f_2)^t$ satisfy

- $f_1 \in H_0^1(\Omega) \cap \mathrm{Dom}(a^{\frac{1}{2}}) \subset L^2(\Omega)$, $f_2 \in L^2(\Omega)$, and
- there exists $m = m(f) > 0$ with

$$\forall \varphi \in C_0^\infty(\Omega) \quad : \quad \left| \int_\Omega (2af_1 + f_2)\overline{\varphi} \mathrm{d}x \right| \leq m\|\nabla\varphi\|.$$

Then there exists $C > 0$ s.t. $\forall t \geq 0$ the unique weak solution $u(t, \cdot)$ of the Cauchy problem satisfies

$$\|u(t,\cdot)\| \leq CI(f)(1+t)^{-1/2},$$

$$\|\partial_t u(t,\cdot)\| + \|\nabla u(t,\cdot)\| \leq CI(f)(1+t)^{-1},$$

where $C > 0$ and

$$I(f)^2 = \|f_1\|^2 + \|\nabla f_1\|^2 + \|f_2\|^2 + m.$$

If, in addition, $\Omega = \mathbb{R}^d$ and $c|x|^\beta \leq a(x) \leq C|x|^\beta$, a.e. $|x| > r_0 > 0$, with $\beta, c, C > 0$, then $\|u(t,\cdot)\| \leq CI(f)(1+t)^{-(1+\beta)/(2+\beta)}$

Remarks - Theorem 2

- More general result than Theorem 1 ($\Omega \subset \mathbb{R}^d$, $a \in L^1_{\mathrm{loc}}(\Omega)$, $d \geq 1$)

- Deduce the L^2 and energy estimates in Theorem 1 from Theorem 2
  - ◇ in particular, the initial data in Theorem 1 satisfies
  $$m \leq C'(\|2af_1 + f_2\|_{L^1} + \|2af_1 + f_2\|).$$
  (use Sobolev inequalities: $d \geq 3$ needed)

- Use different techniques from I-T: we rely on analysis of the resolvent norm of G along $\mathrm{i}\mathbb{R}$ and on the theory of C_0-semigroups

- Uniform positivity ($a(x) \geq a_0 > 0$ a.e. in Ω) yields
$$\sup_{|b| \geq 1} \|(G - \mathrm{i}b)^{-1}\| < \infty$$

- The underlying cause of polynomial decay of solutions is the presence of a singularity at 0
$$\|(G - \mathrm{i}b)^{-1}\| \leq C|b|^{-1}, \quad |b| \to 0$$
driven by the unboundedness of the damping at ∞ ($\sigma_{\mathrm{ess}}(G) \cap \mathrm{i}\mathbb{R} = \{0\}$)
  - ◇ proof requires uniform positivity only outside some ball in Ω

Further Remarks - Non-uniformly-positive damping coefficient I

Cases without undamped trajectories in Ω

- For damping a unbounded at $\infty \rightsquigarrow$ expect $\sup_{|b|\geq 1} \|(G - \mathrm{i}b)^{-1}\| < \infty$ to hold
- If $\Omega = \mathbb{R}$, $0 \leq a \in C^\infty(\mathbb{R})$ and $a(x) \to +\infty$ as $|x| \to +\infty$, then $\sup_{|b|\geq 1} \|(G - \mathrm{i}b)^{-1}\| < \infty$ holds ([Arnal, 2022])
- If $\Omega = \mathbb{R}^d$, $a(x) = |x|^\beta$, $\beta > 0$, then $\sup_{|b|\geq 1} \|(G - \mathrm{i}b)^{-1}\| < \infty$ holds ([Léautaud and Lerner, 2017])
- The claims of Theorem 2 are valid in the 2 cases above

Example with undamped trajectory in Ω

$\Omega = \mathbb{R} \times (-1, 1) \subset \mathbb{R}^2, \; a(x, y) = x^{2n}, \; n \in \mathbb{N}, \; (x, y) \in \Omega$

- notice $a(0, y) = 0$
- $\sigma(G) = (-\infty, 0] \sqcup \bigcup\limits_{j \in \mathbb{N}, k \in \mathbb{N}_0} \{\lambda_{j,k}, \overline{\lambda}_{j,k}\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\}$
- for fixed $k \in \mathbb{N}_0$, $\lambda_{j,k} = \frac{i\pi}{2} j + \mathcal{O}_k(j^{-\frac{n}{n+1}})$, $j \to +\infty$

Further Remarks - Non-uniformly-positive damping coefficient II

Example with undamped trajectory in Ω (ctd.)

- resolvent is unbounded at ±i∞: $\|(G-\mathrm{i}b)^{-1}\| \le C|b|^{\frac{n}{n+1}}$, $|b| \to +\infty$
- at 0 we have: $\|(G-\mathrm{i}b)^{-1}\| \le C|b|^{-1}$, $|b| \to 0$
- but energy decay of solutions for suitable initial data f as in Theorem 2

  $$\|\partial_t u(t,\cdot)\| + \|\nabla u(t,\cdot)\| \le C\tilde{I}(f)(1+t)^{-1}$$

  where $C > 0$ and

  $$\tilde{I}(f)^2 = \|f_1\|^2 + \|\nabla f_1\|^2 + \|f_2\|^2 + m + \|\nabla f_2\|^2 + \|\Delta f_1 - 2af_2\|^2$$
- $\rightsquigarrow$ if we remove the uniform positivity assumption, singularities at ∞ may appear but (if milder than the singularity at 0) the latter still determines the rate of decay of solutions

Methods

Energy estimate of the solution

Theorem 3 ([Batty, Chill, and Tomilov, 2016])

Let $(e^{tG})_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$. Assume that $\sigma(G)\cap i\mathbb{R}=\{0\}$ and let $\alpha\geq 1$. Then

$$\|(G-ib)^{-1}\| = \begin{cases} \mathcal{O}(|b|^{-\alpha}), & |b|\to 0,\\ \mathcal{O}(1), & |b|\to\infty \end{cases} \iff \|e^{tG}G(G-1)^{-1}\| = \mathcal{O}(t^{-\frac{1}{\alpha}}), \quad t\to\infty.$$

- to apply the above theorem, we derive resolvent estimates

$$\|(G-\mathrm{i}b)^{-1}\| \leq C|b|^{-1}, \quad |b|\to 0,$$
$$\|(G-\mathrm{i}b)^{-1}\| < \infty, \quad |b|\to +\infty$$

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Thank you for your attention!